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Permutation branes

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Abstract: N-fold tensor products of a rational CFT carry an action of the permutation group S_N. These automorphisms can be used as gluing conditions in the study of boundary conditions for tensor product theories. We present an ansatz for such permutation boundary states and check that it satisfies the cluster condition and Cardy's constraints. For a particularly simple case, we also investigate associativity of the boundary OPE, and find an intriguing connection with the bulk OPE. In the second part of the paper, the constructions are slightly extended for application to Gepner models. We give permutation branes for the quintic, together with some formulae for their intersections.

Keywords: D-branes, Conformal Field Models in String Theory, Conformal and W Symmetry.

In memory of Sonia Stanciu

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Contents

1. Introductory remarks

Thanks to the pioneering work by Cardy [1, 2] and to further, more recent developments (see e.g. [3]–[8]), we have a reasonably complete understanding of a certain class of boundary conditions for rational conformal field theories. This class is distinguished by an especially simple relation between the form of the bulk partition function and the gluing automorphism Ω that connects left- and right-moving symmetry generators

$$W(z) = \Omega\, \overline{W}(\bar{z}) \qquad \text{for} \quad z = \bar{z}\,.$$

If the rational bulk theory comes with a charge-conjugate modular invariant partition function

$$Z(q,\bar{q}) = \sum_{i \in \mathcal{I}} \chi_i(q)\, \chi_{i^+}(q)^*\,,$$

it is the standard gluing condition $\Omega = \text{id}$ that leads to a simple picture of boundary conditions, basically because all Ishibashi states that can be formed for standard gluing conditions are actually present in the bulk theory. The same is true for general Ω as long as, in the bulk partition function $Z(q,\bar{q})$, each sector i is paired with $\omega^{-1}(i^+)$ instead of the charge-conjugate sector i^+, where $\omega(i)$ labels the representation of the symmetry algebra $\mathcal{W}$ induced in the sector i by the action of Ω. The boundary states associated with this class of boundary conditions are often referred to as "Cardy boundary states".

Non-standard gluing conditions (for bulk theories whose partition function is not of the appropriate ω-type) are of crucial importance in string theory, starting with Dirichlet boundary conditions for free bosons. The non-trivial automorphisms Ω available for the gluing conditions of a rational CFT of course depend on the symmetry algebra $\mathcal{W}$ of the model. But there is a simple and natural construction that yields, from a given rational model, new ones on which universal automorphism groups act — namely taking tensor products of the original "component theory". While this operation alone does not produce very exciting effects in the domain of bulk theories, the structure of associated boundary CFTs is much richer: The set of all boundary conditions for an N-fold tensor product, even those that preserve the full symmetry, is not simply obtained by forming tensor products of boundary conditions for the component theory, but includes boundary conditions associated with gluing automorphisms from the permutation group S_N.

In this paper, we present general formulae for such "permutation boundary states", first focusing on situations where the original RCFT has a charge-conjugate (or, if all sectors are self-conjugate, a diagonal) partition function. The permutation boundary states are still "rational" in the sense that the boundary CFT is covariant under the full symmetry algebra $\mathcal{W}^N$, but we will see that the excitation (or open string) spectra of permutation branes in general look rather different from those of tensor products of Cardy states. Our work generalises studies of permutation branes in $G \times G$ WZW models [9]. On the other hand, our boundary states are special cases of the "conformal walls" that appeared in [10]: while a general conformal wall is nothing but a symmetry-breaking conformal boundary condition for a tensor product model (not necessarily with identical factors), our permutation branes can be viewed as walls with perfect transmission (or reflection) of energy between (or within) the identical component theories.

In the next section, we first specify some notations and convenient assumptions, analyse the set of permutation Ishibashi states and give an ansatz for the permutation boundary states. Then, two types of non-linear constraints are checked for this ansatz, namely the cluster condition in subsection 2.2 and Cardy's constraints in subsection 2.3. The latter contains explicit results for spectra of open strings supported by one permutation brane or stretching between two different ones. Up to now, a third important set of sewing relations, namely associativity of the operator product expansion of boundary fields, can only be discussed in a rather simple case (a specific brane for a two-fold tensor product) — which however features an unexpected connection between the boundary OPE on this permutation brane and the OPE of the component bulk CFT.

The third section is devoted to the construction of permutation branes for a CFT of interest to string theory, namely the Gepner model corresponding to the quintic Calabi-Yau manifold. Gepner models are constructed from tensor products of $N = 2$ superconformal minimal models, and many of them admit an action of a permutation group (S_5 for the quintic). Because of various orbifold-like projections involved in the Gepner construction, the bulk partition functions of these models do not satisfy the assumptions used in section 2, so

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some of the methods presented there have to be adapted. The Gepner projections are such that the Ishibashi state content depends strongly on the details of the models (the relative prime factors in minimal model levels and cycle lengths of the permutation), therefore we restrict ourselves to the quintic when computing explicit expressions for partition functions and intersection forms. From the former, one can read off the spectrum of massless open string states, and they indicate that the new branes are BPS as expected. The intersection forms should set the stage for a geometric interpretation of permutation boundary states. This is, however, left as one of several open problems, which are listed in the concluding section.

2. Permutation branes for rational CFTs with diagonal bulk invariant

We start from a unitary rational CFT on the plane with chiral (left- and right-moving) symmetry algebras $\mathcal{W}_L \simeq \mathcal{W}_R \simeq \mathcal{W}$, and with a charge-conjugate partition function $Z(q,\bar{q}) = \sum_{i\in\mathcal{I}} \chi_i(q)\,\chi_{i^+}(q)^*$ associated with a decomposition $\mathcal{H} = \bigoplus_{i\in\mathcal{I}} \mathcal{H}_i \otimes \mathcal{H}_{i^+}$ of the state space into irreducibles. For notational convenience, let us assume that all sectors are self-conjugate, i.e. that the fusion rules satisfy $N_{ii}^0 = 1$ where 0 denotes the vacuum sector. Furthermore, it will at a certain point be advantageous to assume that no fusion channel with non-trivial multiplicity occurs: $N_{ij}^k = 0$ or 1.

We can then form an N-fold tensor product of this rational CFT, with chiral algebra $\mathcal{W}^N := \mathcal{W} \times \cdots \times \mathcal{W}$ and partition function

$$Z^{(N)}(q,\bar{q}) = (Z(q,\bar{q}))^N = \sum_{I\in\mathcal{I}^N} \chi_I(q)\,\chi_I(q)^*$$

where we have used the multi-index $I := (i_1,\ldots,i_N)$ to label the characters of the tensor product theory, $\chi_I(q) := \chi_{i_1}(q)\cdots\chi_{i_N}(q)$. Again, the partition function $Z^{(N)}(q,\bar{q})$ is a diagonal (and at the same time, because of our simplifying assumption $i = i^+$, a charge conjugate) modular invariant.

The tensor product theory admits outer automorphisms which act by permutations of the components of $\mathcal{W}^N$, namely

$$\Omega_\pi \;:\; W^{[k]}(z) \longmapsto W^{[\pi(k)]}(z)$$

where $\pi \in S_N$ is a permutation and where $W^{[k]}(z) := \mathbf{1} \otimes \cdots \otimes W(z) \otimes \cdots \otimes \mathbf{1}$ denotes the action of a $\mathcal{W}$-generator in the k^{th} component theory, for $k = 1,\ldots,N$. We want to construct boundary conditions for the tensor product theory with Ω_π appearing as a gluing automorphism, i.e. we link left- and right-moving generators by the condition

$$W^{[k]}(z) = \overline{W}^{[\pi(k)]}(\bar{z}) \tag{2.1}$$

along the boundary $z = \bar{z}$ of the upper half-plane. These boundary conditions are conformal since Ω_π leaves the diagonal energy-momentum tensor $T = T^{[1]} + \ldots + T^{[N]}$ fixed. The gluing conditions (2.1) in fact guarantee that the full symmetry algebra $\mathcal{W}^N$ is represented on the Hilbert space of the boundary CFT (with constant boundary condition along the real line) even though the analytic continuation from upper to lower half-plane prescribed by (2.1) is not the standard one.

2.1 An ansatz for permutation boundary states

Permutation boundary states are built up from objects that implement the permutation gluing conditions (2.1). These Ishibashi states can be expanded as

$$|I\rangle\rangle_\pi = \sum_{\boldsymbol{M}} |i_1, M_1\rangle \otimes \cdots \otimes |i_N, M_N\rangle \otimes U|i_{\pi^{-1}(1)}, M_{\pi^{-1}(1)}\rangle \otimes \cdots \otimes U|i_{\pi^{-1}(N)}, M_{\pi^{-1}(N)}\rangle \quad (2.2)$$

where $\boldsymbol{M} = (M_1, \ldots, M_N)$ is used to label orthonormal bases $|i_k, M_k\rangle$ of energy eigenstates in the representations $\mathcal{H}_{i_k}$ of $\mathcal{W}$, and where the operator U in front of the right-movers is the chiral CPT operator as usual. It is important to realize that the objects $|I\rangle\rangle_\pi$ are available only for certain multi-indices $I = (i_1, \ldots, i_N)$: Since the partition function of the bulk theory is diagonal,

$$|I\rangle\rangle_\pi \text{ exists if and only if } i_k = i_{\pi^{-1}(k)} \text{ for all } k = 1, \ldots, N\,. \quad (2.3)$$

This means that the two $\mathcal{W}$-representations i_k, i_l have to coincide whenever k and l are elements of the same cycle of the permutation

$$\begin{aligned}\pi = &\left(n_1^\pi\, \pi(n_1^\pi) \ldots \pi^{\Lambda_1^\pi - 1}(n_1^\pi)\right) \left(n_2^\pi\ \pi(n_2^\pi) \ldots \pi^{\Lambda_2^\pi - 1}(n_2^\pi)\right) \cdots \\ &\cdots \left(n_{P^\pi}^\pi \pi(n_{P^\pi}^\pi) \ldots \pi^{\Lambda_{P^\pi}^\pi - 1}(n_{P^\pi}^\pi)\right). \end{aligned} \quad (2.4)$$

Here, we have chosen an arbitrary element $n_\nu^\pi \in \{1, \ldots, N\}$ as representative of the ν^{th} cycle

$$C_\nu^\pi = \left(n_\nu^\pi\ \pi(n_\nu^\pi) \ldots \pi^{\Lambda_\nu^\pi - 1}(n_\nu^\pi)\right);$$

the n_ν^π will be kept fixed, and we denote the length of C_ν^π by Λ_ν^π, for $\nu = 1, \ldots, P^\pi$, where P^π is the number of cycles of π.

We will abbreviate the condition $i_k = i_{\pi^{-1}(k)}$ on the Ishibashi states by inserting a Kronecker symbol $\delta_I^{C^\pi}$ into the following formulae. Note that, even though $i_k = i_l$, the summation indices M_k, M_l in (2.2) are independent of each other — which in particular makes it clear that the permutation Ishibashi states are not just superpositions of standard ones.

Full-fledged boundary states $\|\boldsymbol{\alpha}\,\rangle\rangle_\pi$ for the gluing conditions (2.1) can be written as linear combinations of the Ishibashi states (2.2),

$$\|\boldsymbol{\alpha}\,\rangle\rangle_\pi = \sum_I\ \delta_I^{C^\pi}\ B_{\boldsymbol{\alpha}}^I\, |I\rangle\rangle_\pi\,. \quad (2.5)$$

We will now present a natural ansatz for the coefficients $B_{\boldsymbol{\alpha}}^I$ and then perform two important consistency checks, involving the so-called cluster condition resp. Cardy's conditions on the strip partition functions.

In our ansatz for the coefficients in (2.5), $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n_P})$ is a multi-index with as many components as there are independent labels $i_{n_\nu^\pi}$ in the Ishibashi states $|I\rangle\rangle_\pi$; each α_ν is taken from the label set $\mathcal{I}$ of all $\mathcal{W}$-representations. The formula for $B_{\boldsymbol{\alpha}}^I$ is multiplicative in the $i_{n_\nu^\pi}$ and reduces to Cardy's solution for the component theory in the special case $N = 1$:

$$B_{\boldsymbol{\alpha}}^I = B_{\alpha_1}^{i_{n_1}} \cdots B_{\alpha_P}^{i_{n_P}} \qquad \text{with} \qquad B_{\alpha_\nu}^{i_{n_\nu}} = \frac{S_{\alpha_\nu\, i_{n_\nu}}}{\left(S_{0\, i_{n_\nu}}\right)^{\frac{\Lambda_\nu}{2}}}\ ; \quad (2.6)$$

we have dropped the superscript $^\pi$ from the cycle representatives n_ν^π, the cycle lengths Λ_ν^π and the number of cycles P^π . As usual, the matrix S implements modular transformation of the $\mathcal{W}$-characters; therefore, the matrix $\big(B_{\boldsymbol{\alpha}}^I\big)$ is invertible, and the ansatz (2.6) will provide a complete set, in the sense of [5], of boundary states for fixed gluing condition Ω_π — provided all sewing constraints are satisfied.

2.2 Cluster condition

The cluster condition gives a first check for the consistency of our ansatz. In contrast to Cardy's constraints, it involves a single boundary condition and genus zero world-sheets only. It is obtained from a sewing relation which compares two different ways (orderings of OPEs) to evaluate a bulk field two-point function in the presence of the boundary (see e.g. [3, 5, 11, 6, 12]), with subsequent projection on the identity channel — which governs the long range behaviour (the clustering properties) of the two-point function. (Strictly speaking, this requires the boundary condition to be 'fundamental' in the sense that only a single vacuum character is present in the open string spectrum; we will see in the next subsection that our permutation branes have this property.) The cluster condition reads

$$B_{\boldsymbol{\alpha}}^I\, B_{\boldsymbol{\alpha}}^J = \sum_K \Xi_{IJK}\; B_{\boldsymbol{\alpha}}^0\, B_{\boldsymbol{\alpha}}^K\,, \tag{2.7}$$

where Ξ_{IJK} is a product $C{\cdot}\mathsf{F}$ involving a structure constant C from the OPE of the two bulk fields $\varphi_{I,I}$ and $\varphi_{J,J}$, as well as a certain element F of the fusing matrix (which relates the conformal blocks in the two channels). Since these data are known explicitly only for few CFTs, the Ξ_{IJK} can usually not be determined directly.

However, for the component theory ($N=1$), a simple expression for the Ξ_{IJK} follows from the Verlinde formula and the fact that Cardy's boundary states provide a complete set of consistent boundary states for standard gluing conditions (see in particular [7, 8]). Plugging in Cardy's solutions for the B_α^i into (2.7) for $N=1$ yields

$$\Xi_{ijk} = \left(\frac{S_{00}S_{0k}}{S_{0i}S_{0j}}\right)^{\frac{1}{2}} N_{ij}^k\,. \tag{2.8}$$

Clearly, the constants Ξ_{IJK} for tensor product bulk fields $\varphi_I = \varphi_{i_1}\otimes\cdots\otimes\varphi_{i_N}$ etc. factorise into N constants $\Xi_{i_l j_l k_l}$. The constraint (2.7) implicitly requires that all the bulk fields involved have non-vanishing one-point functions. In particular, the bulk OPE on the rhs is usually truncated by taking $\langle\,\cdot\,\rangle_\alpha$. Thus, in our situation, only those tensor products occur that obey the $\delta_I^{C^\pi}$-restriction from the Ishibashi states, and we have

$$\Xi_{IJK} = \Big(\Xi_{i_{n_1} j_{n_1} k_{n_1}}\Big)^{\Lambda_1} \cdots \Big(\Xi_{i_{n_{P^\pi}}\, j_{n_{P^\pi}}\, k_{n_{P^\pi}}}\Big)^{\Lambda_{P^\pi}}\,.$$

As a result, the summation in (2.7) splits into P^π independent ones (with each $k_{n_\nu^\pi}$ ranging over the full index set $\mathcal{I}$). Our simplifying assumptions on the fusion rules imply that $(N_{ij}^k)^M = N_{ij}^k$ for all $M\neq 0$; combining this with the fact that the generalised quantum dimensions represent the fusion rules,

$$\frac{S_{ia}}{S_{0a}}\,\frac{S_{ja}}{S_{0a}} = \sum_k\; N_{ij}^k\,\frac{S_{ka}}{S_{0a}}\,, \tag{2.9}$$

it is an easy exercise to check that the boundary states (2.6) indeed satisfy the cluster condition. It appears to be mainly a problem of notation to include non-trivial multiplicities of fusion channels: As in [7, 8], one would have to introduce additional block labels to disentangle summations. In the following, the simplifying property $N_{ij}^k < 2$ will play no role.

2.3 Cardy's conditions

The computations necessary to test Cardy's conditions [2] provide much more interesting physical information than the cluster condition. Cardy's conditions involve pairs of boundary states and require that the quantity

$$Z_{\boldsymbol{\alpha\beta}}(q) := \langle\!\langle\, \boldsymbol{\beta} \,\|\, \tilde{q}^{L_0 - \frac{c}{24}} \,\|\, \boldsymbol{\alpha} \,\rangle\!\rangle$$

can be interpreted as the partition function of a CFT on the strip with boundary conditions $\boldsymbol{\alpha}$ resp. $\boldsymbol{\beta}$ along the two boundaries; this partition function records the spectrum of the boundary CFT, or of excitations of open strings attached to the branes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The actual condition hidden in this statement is that $Z_{\boldsymbol{\alpha\beta}}(q) = \sum_k n^k_{\boldsymbol{\alpha\beta}}\, \chi^\circ_k(q)$ must be a linear combinations of characters $\chi^\circ_k(q)$ for some conformal symmetry algebra $\mathcal{W}^\circ$ with (positive) *integer* coefficients $n^k_{\boldsymbol{\alpha\beta}}$. We use $\tilde{q} = \exp(-2\pi i/\tau)$ and $q = \exp(2\pi i\tau)$, with τ being the modular parameter.

In order to check Cardy's constraints, we first need to compute "sandwiches" of the closed string propagator between two Ishibashi states, possibly corresponding to two different permutations $\pi, \sigma \in S_N$:

$$\begin{aligned}
{}_\sigma\langle\!\langle I' |\, \tilde{q}^{L_0^{\rm tot} - \frac{c^{\rm tot}}{24}}\, | I \rangle\!\rangle_\pi \;=\; & \sum_{\boldsymbol{M},\boldsymbol{M}'} \tilde{q}^{h(i_1,M_1)+\ldots+h(i_N,M_N) - N\frac{c}{24}} \\
& \times \langle i'_1, M'_1 \,|\, i_1, M_1 \rangle \cdots \langle i'_N, M'_N \,|\, i_N, M_N \rangle \\
& \times \langle i_{\pi^{-1}(1)}, M_{\pi^{-1}(1)} | i'_{\sigma^{-1}(1)}, M'_{\sigma^{-1}(1)} \,\rangle \cdots \\
& \langle i_{\pi^{-1}(N)}, M_{\pi^{-1}(N)} | i'_{\sigma^{-1}(N)}, M'_{\sigma^{-1}(N)} \,\rangle
\end{aligned} \tag{2.10}$$

We have pulled out the conformal weights $h(i_k, M_k)$ of the states $|i_k, M_k\rangle$, and c denotes the central charge of the component theory. For the rhs to be non-vanishing, the representation labels and summation indices have to meet the following conditions, for all $k = 1, \ldots, N$:

$$\begin{aligned}
i'_k &= i_k\,, \qquad i_k = i_{\pi(k)}\,, \qquad i_k = i_{\sigma(k)}\,, \qquad i_k = i_{\pi^{-1}\sigma(k)} \\
M'_k &= M_k \qquad M_k = M_{\pi^{-1}\sigma(k)}\,.
\end{aligned} \tag{2.11}$$

The second line means that there are as many free summation indices in (2.10) as the permutation $\pi^{-1}\sigma$ has cycles — namely $P^{\pi^{-1}\sigma}$ in the above notations. For non-vanishing contributions to (10), M_k and M_l are equal if k and l are in the same $(\pi^{-1}\sigma)$-cycle $C_\rho^{\pi^{-1}\sigma}$; as a consequence, the highest weight $h(i_k, M_k)$ appears with a factor $\Lambda_\rho^{\pi^{-1}\sigma}$. The M_k-summation then yields a $\mathcal{W}$-character with argument $\tilde{q}^{\Lambda_\rho^{\pi^{-1}\sigma}}$. After modular transformation to the open string channel, we end up with characters evaluated at fractional powers $q^{1/\Lambda}$.

The restrictions on the representation labels i_k are more severe since they contain the Ishibashi constraints from above. We can summarise them by inserting Kronecker symbols $\delta_{I,I'}$ and $\delta_I^{C^{\pi*\sigma}}$ which mean that the overlap vanishes unless $i_k = i'_k$ and

$$i_k = i_l \quad \text{whenever} \quad l = g(k) \quad \text{for some element} \quad g \in \pi * \sigma := \text{span}\{\pi, \sigma\} \subset S_N\,,$$

the subgroup of S_N generated by π and σ. One can show that $\delta_I^{C^{\pi*\sigma}} = \delta_I^{C^\pi} \cdot \delta_I^{C^\sigma}$ holds. (Below, we will sometimes refer to the orbits $C^{\pi*\sigma}$ of the subgroup $\pi * \sigma$ as "cycles", by slight abuse of terminology.) With these abbreviations, and with the usual normalisation [2] of Ishibashi states, we have

$$\begin{aligned}
{}_\sigma\langle\langle I'|\, \tilde{q}^{L_0-\frac{c}{24}}\, |I\rangle\rangle_\pi &= \delta_{I,I'} \cdot \delta_I^{C^{\pi*\sigma}} \cdot \chi_{i_{n_1}}\left(\tilde{q}^{\Lambda_1}\right) \cdots \chi_{i_{n_P}}\left(\tilde{q}^{\Lambda_P}\right) \\
&= \delta_{I,I'} \cdot \delta_I^{C^{\pi*\sigma}} \cdot \sum_{j_1,\ldots,j_P} S_{i_{n_1}j_1} \cdots S_{i_{n_P}j_P}\, \chi_{j_1}\left(q^{\frac{1}{\Lambda_1}}\right) \cdots \chi_{j_P}\left(q^{\frac{1}{\Lambda_P}}\right). \qquad (2.12)
\end{aligned}$$

In a feeble attempt to avoid cluttering notations completely, we have omitted all superscripts ${}^{\pi^{-1}\sigma}$ here, i.e. $n_\rho = n_\rho^{\pi^{-1}\sigma}$, $\Lambda_\rho = \Lambda_\rho^{\pi^{-1}\sigma}$, $P = P^{\pi^{-1}\sigma}$. Again, the χ_{j_λ} denote characters of the original symmetry algebra $\mathcal{W}$. Note that the cycles of $\pi^{-1}\sigma$ and $\sigma^{-1}\pi$ coincide up to internal reordering, therefore the result is symmetric in π and σ. For $\pi \neq \sigma$, the characters in the last line of (2.12) are familiar from the twisted sectors of cyclic orbifold theories. We will make further remarks on this at the end of the subsection.

Let us briefly look at two special cases before tackling Cardy's conditions in the general situation. For $\sigma = \pi$, the rhs of (2.12) involves a product of $N = P^{\text{id}}$ characters $\chi_{j_k}(q)$ of $\mathcal{W}$, and the cycle restrictions of $\pi * \pi$ are simply those already captured in $\delta_I^{C^\pi}$. Inserting the ansatz (2.6) for the full permutation boundary states, one finds (recall that $S = S^*$ with our assumptions)

$$Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\pi}(q) = \sum_{J=(j_1,\ldots,j_N)} \left[\prod_{\nu=1}^{P^\pi} n_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\pi}^{(\nu)\,J}\right] \chi_{j_1}(q) \cdots \chi_{j_N}(q) \qquad (2.13)$$

with

$$n_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\pi}^{(\nu)\,J} = \sum_{i_{n_\nu} \in \mathcal{I}} \frac{S_{\alpha_\nu i_{n_\nu}} S_{\beta_\nu i_{n_\nu}}}{\left(S_{0\, i_{n_\nu}}\right)^{\Lambda_\nu}} \prod_{k \in C_\nu^\pi} S_{i_{n_\nu} j_k}\,.$$

The numbers $n_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\pi}^{(\nu)\,J}$ can be calculated with the help of the quantum dimension property (2.9) and of the Verlinde formula:

$$\begin{aligned}
n_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\pi}^{(\nu)\,J} &= \sum_{k_1,\ldots,k_{\Lambda_\nu-1}} N_{j_{n_\nu}\, j_{\pi(n_\nu)}}^{k_1} N_{k_1\, j_{\pi^2(n_\nu)}}^{k_2} \cdots N_{k_{\Lambda_\nu-2}\, j_{\pi^{\Lambda_\nu-1}(n_\nu)}}^{k_{\Lambda_\nu-1}} \sum_{i_{n_\nu}} \frac{S_{\alpha_\nu\, i_{n_\nu}} S_{\beta_\nu\, i_{n_\nu}} S_{k_{\Lambda_\nu-1}\, i_{n_\nu}}}{S_{0\, i_{n_\nu}}} \\
&= \left(\prod_{k \in C_\nu^\pi} N_{j_k}\right)_{\alpha_\nu \beta_\nu}\,, \qquad (2.14)
\end{aligned}$$

where we have used associativity of the fusion rules in the last step. All in all, we obtain the following partition function for two boundary states associated with the same permutation automorphism:

$$Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\pi}(q) = \sum_{j_1,\dots,j_N} \prod_{\nu=1}^{P^\pi} \left(\prod_{k \in C_\nu^\pi} N_{j_k} \right)_{\alpha_\nu \beta_\nu} \chi_{j_1}(q) \cdots \chi_{j_N}(q) \,. \tag{2.15}$$

This is a sum of $\mathcal{W}^N$-characters with coefficients given by the α_ν-β_ν-entries of "cycle-wise" products of fusion matrices $(N_i)_{jk} = N_{ij}^k$ from the component theory. Note that our permutation boundary states are "orthonormal" in the sense that the vacuum occurs (once) in the overlap (2.15) if and only if $\boldsymbol{\alpha}_\pi = \boldsymbol{\beta}_\pi$.

The case $\sigma = \mathrm{id} \neq \pi$ involves different products of $\mathcal{W}$-characters, but is still easy to handle because $\delta_I^{C^{\pi^{-1}\sigma}} = \delta_I^{C^{\pi * \sigma}} = \delta_I^{C^\pi}$ holds. The boundary states $\|\boldsymbol{\beta}\rangle\rangle_{\mathrm{id}}$ are tensor products of Cardy boundary states for the $\mathcal{W}$-theory, but the $\delta_{I,I'}$-projection in the Ishibashi overlap (2.12) means that only those $|I'\rangle\rangle_{\mathrm{id}}$ contribute to the partition function that obey $i'_k = i'_{\pi(k)}$. Using this, eq. (2.12) and our ansatz (2.6), we get

$$Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_{\mathrm{id}}}(q) = \sum_{j_1,\dots,j_P} \left[\prod_{\nu=1}^{P} n^{(\nu)\, j_\nu}_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_{\mathrm{id}}} \right] \chi_{j_1}\left(q^{\frac{1}{\Lambda_1}}\right) \cdots \chi_{j_P}\left(q^{\frac{1}{\Lambda_P}}\right),$$

with $P = P^\pi$, $\Lambda_\nu = \Lambda_\nu^\pi$, and with similar multiplicities as before:

$$n^{(\nu)\, j_\nu}_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_{\mathrm{id}}} = \sum_{i_{n_\nu}} \frac{S_{\alpha_\nu i_{n_\nu}} S_{i_{n_\nu} j_\nu}}{\left(S_{0\, i_{n_\nu}}\right)^{\Lambda_\nu}} \prod_{k \in C_\nu^\pi} S_{\beta_k i_{n_\nu}} \,.$$

As for the power in the denominator, the ν^{th} π-cycle contributes $\Lambda_\nu/2$ and each of the Λ_ν id-"cycles" contributes $1/2$. The S-matrix relations used above now yield the expression

$$Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_{\mathrm{id}}}(q) = \sum_{j_1,\dots,j_P} \left[\prod_{\nu=1}^{P} \left(\prod_{k \in C_\nu^\pi} N_{\beta_k} \right)_{\alpha_\nu j_\nu} \right] \chi_{j_1}\left(q^{\frac{1}{\Lambda_1}}\right) \cdots \chi_{j_P}\left(q^{\frac{1}{\Lambda_P}}\right). \tag{2.16}$$

As a side-remark, we observe that the "vacuum Cardy state" $\|\mathbf{0}\rangle\rangle_{\mathrm{id}}$ of the tensor product theory provides a projection on single $\mathcal{W}^{P^\pi}$-characters in the sense that

$$Z_{\boldsymbol{\alpha}_\pi \mathbf{0}_{\mathrm{id}}}(q) = \chi_{\alpha_1}\left(q^{\frac{1}{\Lambda_1}}\right) \cdots \chi_{\alpha_{P^\pi}}\left(q^{\frac{1}{\Lambda_{P^\pi}}}\right). \tag{2.17}$$

Therefore, any boundary state $\|A\rangle\rangle_\pi$ for the gluing conditions (2.1) that is compatible with $\|\mathbf{0}\rangle\rangle_{\mathrm{id}}$ lies in the lattice cone spanned by the $\|\boldsymbol{\alpha}\rangle\rangle_\pi$.

We have shown that $Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q)$ meets Cardy's conditions for $\sigma = \pi$ and for $\sigma = \mathrm{id}$. In the general case, the partition function is of the form

$$Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q) = \sum_{J=(j_1,\dots,j_P)} \left[\prod_{\lambda=1}^{P^*} n^{(\lambda)\, J}_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma} \right] \chi_{j_1}\left(q^{\frac{1}{\Lambda_1}}\right) \cdots \chi_{j_P}\left(q^{\frac{1}{\Lambda_P}}\right) \tag{2.18}$$

where now P and the Λ_ρ refer to number and lengths of the cycles of $\pi^{-1}\sigma$, while P^* denotes the number of "cycles" or $C_\nu^{\pi * \sigma}$ of $\pi * \sigma$: The Kronecker symbols in (2.12) imply that precisely P^* of the i_{n_ν} are independent, while the constraints on the summation indices M_k in (2.11) leave us with a product of P characters.

The slightly lengthy proof for the fact that the coefficients $n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}$ are indeed positive integers is given in appendix A. All in all, we may conclude that the partition functions $Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q)$ satisfy Cardy's conditions for all permutations $\pi, \sigma \in S_N$ as long as the boundary state coefficients are given by formula (2.6). Furthermore, any other (compatible) permutation boundary state lies in the integer lattice over these states.

Some comments on the form (2.18) of $Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q)$ are in order. These partition functions describe spectra of boundary fields, more specifically, if $\boldsymbol{\alpha}_\pi \neq \boldsymbol{\beta}_\sigma$, spectra of boundary condition changing operators (BCCOs). Already for the case $\pi = \sigma \neq \mathrm{id}$, the spectra in (2.13), (2.14) are in general different from those obtained with tensor products of component Cardy states, because of the cycle-wise products of fusion matrices in (2.14).

That the partition functions for $\pi \neq \sigma$ are built from characters $\chi_j(q^{1/\Lambda_\nu})$ can be understood as follows: While both gluing automorphisms Ω_π and Ω_σ preserve the full symmetry algebra $\mathcal{W}^N$, it is a priori only the subalgebra $\mathcal{W}^\circ$ with $\Omega_\pi(A) = \Omega_\sigma(A)$ for all $A \in \mathcal{W}^\circ$, i.e. the fixed-point algebra under $\Omega_{\pi^{-1}\sigma}$, that is represented on BCCOs which mediate between the two gluing conditions. The form of this subalgebra is determined (up to isomorphism) by the cycle lengths Λ_ν of $\pi^{-1}\sigma$, i.e. by the conjugacy class of this permutation. We have $cW^\circ \simeq C_{\Lambda_1}\mathcal{W} \times \cdots \times C_{\Lambda_P}\mathcal{W}$ where $C_M\mathcal{W}$ consists of all elements of $\mathcal{W}^M$ that are invariant under the cyclic permutation of order M, i.e. $C_M\mathcal{W}$ is the observable algebra of a cyclic $\mathbb{Z}_M$-orbifold, see [13]. This reference also shows how the characters $\chi_j\left(q^{\frac{1}{M}}\right)$ enter the partition function of $\mathbb{Z}_M$ cyclic orbifolds: To construct such a partition function, one first projects the M-fold tensor product space onto invariant states; the states left fixed by $\mathbb{Z}_M$ contribute $Z(q^N, \bar{q}^N)$, where as before Z belongs to the given component theory. To ensure modular invariance, one has to add twisted sectors, arising from these "fixed points" after action with modular group generators [13]; this produces characters with fractional powers of q. One can express

$$\chi_j\left(q^{\frac{1}{M}}\right) = \sum_{s=0}^{M-1} \chi_{\widehat{(j,s)}}(q) \tag{2.19}$$

as a sum of cyclic orbifold characters $\chi_{\widehat{(j,s)}}(q)$ corresponding to twisted sectors labelled by s; see [14] for more details. The highest weights on the rhs of (2.19) are computed with L_0 from the cyclic orbifold model and read [13, 14]

$$h_{(j,s)} = \frac{h_j + s}{M} + \frac{M^2 - 1}{M}\,\frac{c}{24} \tag{2.20}$$

where h_j and c are conformal dimensions resp. central charge of the given component theory.

The decomposition (2.19) now fits with general expectations: Computation of open string partition functions $Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q)$ involves performing a modular transformation of traces of $\tilde{q}^{H^c}\,\Omega_{\pi^{-1}\sigma}$, where H^c is the closed string hamiltonian. Indeed, this is one way to compute the partition function in the $(\pi^{-1}\sigma)$-twisted sector of cyclic orbifolds; see also [15] for more details.

The symmetry algebra of a whole system $\boldsymbol{\alpha}^i_{\pi_i}$, $i = 1, 2, \ldots$, of permutation branes is the fixed point algebra $\left(\mathcal{W}^N\right)^{\mathcal{F}}$ with the subgroup $\mathcal{F} = \langle\, \pi_i^{-1}\pi_j \,|\, i,j = 1, \ldots \rangle \subset S_N$.

Characters and fusion rules of such general permutation orbifolds have been studied in detail in Bantay's works [16]. If all possible permutation branes are included into the system, then $\mathcal{F} = S_N$ and we arrive at the symmetry algebra of a symmetric product. Thus, permutation branes have "long" open strings ending on them. In contrast to the setup in [17], where boundary states for closed strings in a symmetric product background were constructed, here it is permutation gluing automorphisms that introduce long open strings into a theory of ordinary ("short") closed strings.

2.4 Boundary OPE in a special case

Beyond the cluster condition and Cardy's conditions, there are of course further sewing relations which have to be satisfied in a consistent boundary CFT [3]. Most notably, the structure constants in the OPE of boundary fields are constrained by associativity. Solutions for the structure constants in terms of the fusing matrix have been worked out by Runkel [6] and later by other authors in [7, 8]. The formulae there apply to any unitary rational CFT (with diagonal bulk partition function) as long as Cardy-type boundary states for standard gluing conditions ($\Omega = \mathrm{id}$) are used. Thus, we can unfortunately not carry them over to our more complicated situation of permutation branes. We expect that a complete solution for the boundary OPE can be given once fusing matrices for permutation orbifolds are known.

For the time being, let us merely have a brief look at a simple special case, which already displays some interesting features. We restrict to a twofold tensor product with $\pi = (1\,2)$ and focus at the brane $\|\boldsymbol{\alpha}\rangle\rangle_\pi = \|\mathbf{0}\rangle\rangle_\pi$. According to (2.13), (2.14), the boundary spectrum for this special boundary condition is given by

$$Z_{\mathbf{0}_\pi\,\mathbf{0}_\pi}(q) = \sum_{j\in\mathcal{I}} \chi_j(q)\,\chi_j(q)\,. \tag{2.21}$$

It "coincides" with that of the original bulk theory if we "identify" the right-moving charges of bulk fields with the second tensor factors of (chiral) boundary fields. (If we drop our assumption of self-conjugate sectors, (2.21) will be replaced by the charge-conjugate bilinear and some of the following arguments have to be adapted accordingly.)

This observation allows to solve the associativity condition for the OPE of boundary fields supported by $\|\mathbf{0}\rangle\rangle_\pi$: That condition has the schematic form (for a self-conjugate theory)

$$\sum_P C_{IJP}\,C_{KLP}\mathsf{F}_{PQ} = C_{JKQ}\,C_{IQL}\,;$$

the C are the OPE coefficients in question, indexed by $I = (i,i)$, $J = (j,j)$ etc. as counted in (2.21), and F is the fusing matrix for the conformal blocks of the boundary CFT. In the special case at hand, the tensor product structure of the boundary fields implies that this fusing matrix is just the square of the fusing matrix of the component theory. But this means that the sewing relation for the boundary fields in (2.21) has the same form as the one for the bulk fields in the component theory. Therefore, the coefficients in the *bulk* OPE of the original component theory provide a solution to the sewing relations of the *boundary* OPE for the boundary condition $\|\mathbf{0}\rangle\rangle_\pi$ in the two-fold tensor product theory!

We have not analysed sewing relations involving BCCOs yet, but assuming that the above OPE coefficients survive all further tests, one can in particular conclude that the boundary fields supported by $\|\mathbf{0}\rangle\rangle_\pi$ are mutually local. Namely, the bulk OPE coefficients ensure analyticity of the correlators. In view of the role of boundary OPEs for the analysis of non-commutative behaviour of branes [18], this suggests that the (low-energy) world-volumes of the branes $\|\mathbf{0}\rangle\rangle_\pi$ are commutative spaces, whatever the underlying component theory is. If a sigma-model interpretation of the original theory is available (with target M say), then $\|\mathbf{0}\rangle\rangle_\pi$ should describe a brane whose world-volume is the diagonal in $M \times M$. This is confirmed by the results obtained in [9, 19] for WZW models with group target $G \times G$, using the classical interpretation of WZW gluing conditions resp. analysing the module structure of the algebra of boundary fields in the infinite level limit.

3. Permutation branes for the quintic

Gepner models provide an important class of string backgrounds involving tensor products of rational CFTs [20]. They were probably the first theories where cyclic permutation orbifolds of bulk CFTs were studied in the literature [13, 21]. In this context, the orbifold construction yields new closed string backgrounds, which correspond to new Calabi-Yau manifolds and to new spectra of massless closed (or heterotic) string states. Here, we want to use permutation gluing to obtain new (rational) branes for a given closed string background, which remains unaltered.

The partition functions of Gepner models are built from tensor product characters [20]

$$\chi^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(q) := \chi_{s_0}(q)\, \chi^{l_1}_{m_1,s_1}(q) \cdots \chi^{l_r}_{n_r,s_r}(q)$$

where each $\chi^{l_j}_{m_j,s_j}(q)$ is a character of the $N = 2$ super Virasoro algebra with level k_j — more precisely of the bosonic subalgebra (hence the additional label s_j). The k_j, $j = 1, \ldots, r$, are chosen in such a way that the central charges $c_j = 3k_j/(k_j+2)$ add up to 3, 6 or 9, corresponding to string compactifications down to $D = 8$, 6 resp. 4 external dimensions. $\chi_{s_0}(q)$ is a character of the $d = D-2$ free fermions associated with the transverse external directions. Our notations are as in [20], in particular $\boldsymbol{\mu} = (s_0; m_1, \ldots, m_r; s_1 \ldots, s_r)$ with $s_0, s_j = 0, \ldots, 3$, with $m_j = 0, \ldots, 2k_j + 3$ and with $l_j = 0, \ldots, k_j$. The combinations $l_j + m_j + s_j$ must be even.

Full Gepner model partition functions (those associated with SU(2) modular invariants of type A) have the form

$$Z_{\text{Gep}}(q, \bar{q}) \sim \sum_{b_0, b_j} \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\beta} (-1)^{s_0}\, \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(q)\, \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}+b_0\boldsymbol{\beta}_0+b_1\boldsymbol{\beta}_1+\ldots+b_r\boldsymbol{\beta}_r}(\bar{q}) \tag{3.1}$$

where $\boldsymbol{\beta}_0$ is the $(2r+1)$-vector with all entries equal to 1, while $\boldsymbol{\beta}_j$ has zeroes everywhere except for the first and the $(r+j+1)^{\text{st}}$ entry which are equal to 2. The superscript $^{\beta}$ abbreviates Gepner's "β-constraints" in the summation, which implement the GSO projection and ensure that all left-moving states are taken only from the NS sectors of the minimal models or only from the R sectors, see [20]. We have $b_j = 0, 1$ and $b_0 = 0, \ldots, K-1$ with $K := \text{lcm}(4, 2k_j + 4)$.

After some earlier general considerations in [22], explicit boundary states for Gepner models were constructed in [23]. The work [24] then introduced methods to relate abstract CFT boundary conditions to supersymmetric cycles in the Calabi-Yau regime, based on a computation of the intersection form at the Gepner point. This has triggered many interesting developments, see e.g. [25] for a very incomplete list of references, and has led to proposals for a new picture of branes (and bundles) on Calabi-Yau spaces [26].

All these investigations are based on Gepner model boundary states where the full tensor product symmetry algebra is preserved by imposing, in each of the r minimal models individually, A- or B-type gluing conditions, which corresponds to choosing the gluing automorphism $\Omega^{(j)} = \Omega_{\text{mirror}}$ resp. $\Omega^{(j)} = \text{id}$ for all $j = 1, \ldots, r$.

Whenever some of the levels k_j coincide, a permutation group acts on the Gepner model, so one can look for permutation branes — which will still be rational and will still fall into the A- and B-type classification of [22], as the diagonal $N = 2$ algebra is invariant under Ω_π. Unfortunately, the prescription from section 2 cannot be used to construct new boundary states for Gepner models: The bulk partition functions (3.1) are neither diagonal nor charge-conjugate, due to the β-shifts in the right-moving charges, and therefore already the restrictions on representation indices, which select the available π-Ishibashi states, are different from those stated in (2.3) above. Thus we have to adapt our methods to the Gepner case.

It will turn out that the set of admissible Ishibashi states depends strongly on the relative divisibility properties of minimal model levels and cycle lengths of the permutation π. It appears, therefore, that permutation branes for Gepner models can only be constructed case by case, and we will more or less from the start focus on the quintic $(3)^5$, which is the model studied in greatest detail in the literature. It has the added bonus that we need not worry about fixed point resolutions as discussed in [27, 28].

In the next subsection, we will write down permutation boundary states for the quintic that satisfy A-type gluing conditions on the diagonal $N = 2$ algebra, as well as compute the associated partition functions and intersection forms; then we will turn towards B-type permutation branes in subsection 3.2. Formulae expressing some of the intersection forms in terms of charge symmetry generators are collected in appendix B.

3.1 A-type boundary states

Let us first determine which (A-type) Ishibashi states exist for a given permutation $\pi \in S_5$. An A-type π-Ishibashi state for the quintic can be formed iff the left- and right-moving representation labels (l_j, m_j, s_j) resp. $(l_j, m_j + b_0, s_j + b_0 + 2b_j)$ satisfy

$$l_j = l_{\pi(j)}\,, \qquad m_j \equiv m_{\pi(j)} + b_0 \pmod{2k+4}\,, \qquad s_j \equiv s_{\pi(j)} + b_0 + 2b_{\pi(j)} \pmod 4 \tag{3.2}$$

for all $j = 1, \ldots, 5$ and for some choice of b_0 and b_j. In addition, since the external label s_0 is not affected by the permutation, A-type gluing requires that

$$s_0 \equiv s_0 + b_0 + 2\sum_j b_j \pmod 4 \tag{3.3}$$

so that in particular b_0 must be even. The relations on the SU(2) labels l_j in (3.2) are precisely as in the general RCFT setting discussed before — and we anticipate that, in the SU(2) part, the same products of fusion matrices as in (2.15) will show up in partition functions and intersection forms. All further complications arise from the additional summation indices b_0, b_j, which are constrained by the above equations, and by

$$\begin{aligned} \Lambda_\nu b_0 &\equiv 0 \quad (\mathrm{mod}\ 2k+4)\,, \\ \Lambda_\nu b_0 + 2\sum_{j\in C_\nu^\pi} b_j &\equiv 0 \quad (\mathrm{mod}\ 4) \end{aligned} \tag{3.4}$$

for all $\nu = 1, \ldots, P^\pi$ simultaneously; the latter relations are obtained by applying (3.2) repeatedly until the π-cycle C_ν^π of length Λ_ν is closed.

As soon as π has a cycle of length one (i.e. a fixed point), the first condition in (3.4) only leaves the possibilities $b_0 = 0$ or $b_0 = 2k+4$ so that, because of the label periodicities, there is just one independent m-value per π-cycle. The same is true for certain other permutations π and certain other levels, e.g., if we specialise to $k = 3$, for π with $P^\pi = 2$ and $\Lambda_1 = 2$, $\Lambda_2 = 3$. For permutations in the conjugacy class of $\pi = (1\,2\,3\,4\,5)$, on the other hand, the first condition in (3.4) admits all even $b_0 = 0, 2, \ldots, 18$.

As for the constraints on the summation variables b_j, it is not hard to see, by a case-by-case analysis for the quintic, that there is just enough freedom left to render all five s_j-labels independent (apart from the β-constraints).

We summarise the free labels admitted by the Ishibashi constraints for permutation A-type gluing conditions in the case of the quintic. Although only the conjugacy class of π will enter partition functions and intersection forms for two boundary states associated with the same gluing automorphism, it is more convenient to give a list for specific representatives:

$$\begin{array}{ll} \pi = \mathrm{id}: & I = (s_0; l_1, l_2, l_3, l_4, l_5; m_1, m_2, m_3, m_4, m_5; s_j) \\ \pi = (1)(2)(3)(4\,5): & I = (s_0; l_1, l_2, l_3, l_4, l_4; m_1, m_2, m_3, m_4, m_4; s_j) \\ \pi = (1)(2\,3)(4\,5): & I = (s_0; l_1, l_2, l_2, l_3, l_3; m_1, m_2, m_2, m_3, m_3; s_j) \\ \pi = (1)(2)(3\,4\,5): & I = (s_0; l_1, l_2, l_3, l_3, l_3; m_1, m_2, m_3, m_3, m_3; s_j) \\ \pi = (1)(2\,3\,4\,5): & I = (s_0; l_1, l_2, l_2, l_2, l_2; m_1, m_2, m_2, m_2, m_2; s_j) \\ \pi = (1\,2)(3\,4\,5): & I = (s_0; l_1, l_1, l_2, l_2, l_2; m_1, m_1, m_2, m_2, m_2; s_j) \\ \pi = (1\,2\,3\,4\,5): & I = (s_0; l_1, l_1, l_1, l_1, l_1; m_1, m_1+2n, m_1+4n, m_1+6n, m_1+8n; s_j) \\ & \qquad with n = 0, 1, 2, 3, 4\,. \end{array} \tag{3.5}$$

To obtain an ansatz for the full boundary states, we combine the expressions for Gepner branes from [23] with formula (2.6) for permutation branes, and write

$$\|\boldsymbol{\alpha}\,\rangle\!\rangle_{A,\pi} \equiv \|S_0; L_\nu, M_\nu, S_j\,\rangle\!\rangle_{A,\pi} = \frac{1}{\kappa_\alpha^{A,\pi}} \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{\beta} B_{\boldsymbol{\alpha}_{A,\pi}}^{\boldsymbol{\lambda},\boldsymbol{\mu}}\, |\boldsymbol{\lambda},\boldsymbol{\mu}\rangle\!\rangle_{A;\pi}\,. \tag{3.6}$$

We have introduced one boundary state label per independent Ishibashi degree of freedom here: For π with two or more cycles, we use one L_ν-label and one M_ν-label per π-cycle, along with labels S_0 and S_j for $j = 1, \ldots, 5$. For the case $\pi = (1\,2\,3\,4\,5)$, one could start with five labels M_j, but it turns out that these boundary states depend only on the two quantities

$M := \sum_j M_j \pmod{10}$ and $M' := M_2 + 2M_3 + 3M_4 + 4M_5 \pmod 5$. Note that the same (M, M')-labelling also occurred in [29] in connection with a $\mathbb{Z}_5$-orbifold of the quintic.

The coefficients B in (3.6) are given by

$$B^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\alpha}_{A,\pi}} = (-1)^{\frac{s_0^2}{2}} e^{-i\pi\frac{s_0 S_0}{2}} e^{-\frac{i\pi}{2}\sum_{j=1}^5 s_j S_j} \left[\prod_{\nu=1}^{P^\pi} \frac{\sin\pi\frac{(l_\nu+1)(L_\nu+1)}{5}}{\left(\sin\pi\frac{l_\nu+1}{5}\right)^{\Lambda_\nu/2}} e^{i\pi\frac{m_\nu M_\nu}{5}} \right] \tag{3.7}$$

if $\pi \in S_5$ has two or more cycles; for $\pi = (1\,2\,3\,4\,5)$, we use the formula

$$B^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\alpha}_{A,\pi}} = (-1)^{\frac{s_0^2}{2}} e^{-i\pi\frac{s_0 S_0}{2}} e^{-\frac{i\pi}{2}\sum_{j=1}^5 s_j S_j} \frac{\sin\pi\frac{(l_1+1)(L_1+1)}{5}}{\left(\sin\pi\frac{l_1+1}{5}\right)^{5/2}} e^{\frac{i\pi}{5}(m_1 M + 2nM')}$$

where n is the additional label showing up in the Ishibashi states for $\pi = (1\,2\,3\,4\,5)$.

Due to the constraints on the (l_j, m_j, s_j) allowed in Gepner models, the boundary state labels must satisfy $L_\nu + M_\nu + S_j \equiv 0 \pmod 2$ for all $j \in C^\pi_\nu$. Furthermore, labels (L_ν, M_ν, S_j) are identified with $(k-L_\nu, M_\nu+k+2, S_j+2)$ for all $j \in C^\pi_\nu$; for S_5-permutations with a single cycle, we identify (L, M, M', S_j) with $(k-L, M+5, M', S_j+2)$. We restrict ourselves to even S_j.

Now let us compute the partition functions $Z^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}(q)$ for two branes belonging to the same permutation. This is straightforward using Lagrange multipliers ρ_0, ρ_j to disentangle the summations as in [23]. We first introduce the abbreviation

$$f_{s,\rho} := (-1)^{s_0'+S_0-\tilde{S}_0} \sum_{\rho_j=0,1} \delta^{(4)}_{s_0'+S_0-\tilde{S}_0+\rho_0+2\Sigma_j\rho_j+2} \prod_{j=1}^5 \delta^{(4)}_{s_j'+S_j-\tilde{S}_j+\rho_0+2\rho_j}$$

and, for simplicity, neglect the overall normalisation in the following; it can be fixed from Cardy's constraints as in [23, 24]. Then the results for the partition functions are

$$\begin{aligned} Z^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}(q) \sim & \sum_{\boldsymbol{\lambda}',\boldsymbol{\mu}'}^{\mathrm{ev}} \sum_{\rho_0=0}^{19} f_{s,\rho} \times \\ & \times \left[\prod_{\nu=1}^{P^\pi} \left(\prod_{n\in C^\pi_\nu} N_{l_n'} \right)_{L_\nu \tilde{L}_\nu} \delta^{(10)}_{\Sigma_{n\in C^\pi_\nu} m_n' \; + M_\nu - \tilde{M}_\nu + \Lambda_\nu\rho_0} \right] \chi^{\boldsymbol{\lambda}'}_{\boldsymbol{\mu}'}(q) \end{aligned} \tag{3.8}$$

for $P^\pi > 1$. The superscript $^{\mathrm{ev}}$ indicates that $l_j'+m_j'+s_j'$ must be even. For the permutation $\pi = (1\,2\,3\,4\,5)$, the Kronecker symbol in (3.8) is to be replaced by

$$\delta^{(10)}_{m_1'+\ldots+m_5'+M-\tilde{M}+5\rho_0} \; \delta^{(5)}_{m_2'+2m_3'+3m_4'+4m_5'+M'-\tilde{M}'} \,. \tag{3.9}$$

Comparing to the partition functions for ordinary "un-permuted" A-type branes [23], we observe that now products of fusion matrices occur, as in the diagonal RCFT case discussed in section 2. Moreover, the Kronecker restrictions on the labels m_j' of the characters $\chi^{\boldsymbol{\lambda}'}_{\boldsymbol{\mu}'}(q)$ are different from those for $\pi = \mathrm{id}$. By spelling out the partition functions explicitly, one can check that the permutation branes in particular support new spectra of massless open string states. On the other hand, the expressions (3.8) enjoy stability and space-time supersymmetry just as those for $\pi = \mathrm{id}$.

In the intersection form $I^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}$, similar building blocks as in (3.8) show up. We use the definition [30, 24]

$$I^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi} = \mathrm{tr}_{\mathcal{H}_\mathrm{R}}(-1)^F q^{L_0-\frac{c}{24}} = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathrm{RR}}^{\beta} B^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\alpha}_\pi} B^{\boldsymbol{\lambda},-\boldsymbol{\mu}}_{\boldsymbol{\alpha}_\pi}{}_{A;\pi}\langle\langle\boldsymbol{\lambda},\boldsymbol{\mu}|\,(-1)^{F_L}\tilde{q}^{L_0-\frac{c}{24}}\,|\boldsymbol{\lambda},\boldsymbol{\mu}\rangle\rangle_{A;\pi}$$

with $(-1)^{F_L} := (-1)^{J_0^{\mathrm{int}}+\frac{d'}{2}}$, where J_0^{int} is the charge in the internal sector and $d' := 4-\frac{d}{2} = \frac{1}{2}\,d^{\mathrm{int}}$. Trace and summation run over Ramond sectors only. Proceeding along the lines of [24] — exploiting β-constraints, field identifications and the fact that only R ground states (i.e. primaries with labels $(l'_j, l'_j+1, 1)$ or $(l'_j, -l'_j-1, -1)$) contribute to this index — we arrive at

$$I^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi} \sim \sum_{m'_1,\dots,m'_5;\rho_0=0}^{9} \left[\prod_{\nu=1}^{P^\pi}\left(\prod_{n\in C^\pi_\nu} N_{m'_n-1}\right)_{L_\nu\tilde{L}_\nu} \delta^{(10)}_{\Sigma_{n\in C^\pi_\nu} m'_n\ +M_\nu-\tilde{M}_\nu+\Lambda_\nu(2\rho_0+1)}\right]. \tag{3.10}$$

The intersection form for the one-cycle case is obtained by the same replacement of the Kronecker symbol as in (3.8) and (3.9).

In order to save signs in (3.10), we have used the extension $N^{-l-2}_{L\tilde{L}} = -N^{l}_{L\tilde{L}}$ and $N^{-1}_{L\tilde{L}} = N^{k+1}_{L\tilde{L}} = 0$ of the SU(2) fusion rules to "spins" l beyond $0,\dots,k$, see [24]; analogously if two or three spins are in the extended range.

As we have seen already for diagonal RCFTs in subsection 2.3, partition functions $Z^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\sigma}(q)$ for two different permutations depend very much on the relative "location" of the cycles of π and σ. We could compute such partition functions and intersection forms in a relatively straightforward fashion using the formulae from above, but we restrict ourselves to the special case $\sigma = \mathrm{id}$ here.

When evaluating overlaps between π- and id-Ishibashi states, one observes (similarly to the previous general considerations) that further constraints on the labels arise. Obviously, the (independent) l_j and m_j labels from the id-states have to match their (constrained) partners from the π-states. Moreover, the overlaps are non-zero only for specific choices of s_j labels (this follows as in (2.11)), although existence of Ishibashi states alone imposes no relations on these labels. All in all, one is left with precisely one independent (except for β-constraints) array (l_ν, m_ν, s_ν) per π-cycle to sum over.

Writing down partition functions and intersection forms is therefore rather easy. For the spectra of boundary condition changing operators between quintic A-type branes we obtain

$$\begin{aligned} Z^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_{\mathrm{id}}}(q) \sim \sum_{\boldsymbol{\lambda}',\boldsymbol{\mu}'}^{\mathrm{ev}} \sum_{\rho_0=0}^{19}\sum_{\rho_\nu=0,1} (-1)^{s'_0+S_0-\tilde{S}_0}\,\delta^{(4)}_{s'_0+S_0-\tilde{S}_0+\rho_0+2\,\Sigma_\nu\rho_\nu+2} \times \\ \times\left[\prod_{\nu=1}^{P^\pi}\left(\prod_{j\in C^\pi_\nu} N_{\tilde{L}_j}\right)_{L_\nu l'_\nu} \delta^{(10)}_{m'_\nu+M_\nu-\Sigma_{j\in C^\pi_\nu}\tilde{M}_j+\Lambda_\nu\rho_0} \times\right. \\ \left.\times\ \delta^{(4)}_{s'_\nu+\Sigma_{j\in C^\pi_\nu}(S_j-\tilde{S}_j)+\Lambda_\nu\rho_0+2\rho_\nu}\right]{}^{[\pi]}\chi^{\boldsymbol{\lambda}'}_{\boldsymbol{\mu}'}(q) \end{aligned} \tag{3.11}$$

with the expected combinations of cyclic orbifold characters

$$ {}^{[\pi]}\chi^{\boldsymbol{\lambda}'}_{\boldsymbol{\mu}'}(q) = \chi_{s_0}(q)\chi_{(l'_1,m'_1,s'_1)}\left(q^{\frac{1}{\Lambda_1}}\right)\cdots\chi_{(l'_P,m'_P,s'_P)}\left(q^{\frac{1}{\Lambda_P}}\right). $$

Let us have a closer look at the $\delta^{(4)}$-constraint in the last line of (3.11): It prevents characters with s'_ν odd from contributing to the partition function whenever π has a cycle C^π_ν of even length Λ_ν. This does not mean, however, that strings stretching between an ordinary and a π-brane for such a π do not have a Ramond sector. One has to recall[1] that the modes of $\mathbb{Z}_{\Lambda_\nu}$-twisted R-generators are shifted by integer multiples of $1/\Lambda_\nu$, so that for even Λ_ν the minimal model characters with even s'_ν may actually belong to twisted R-representations. The same effect has to be taken into account when computing the intersection form between a π-brane and an ordinary A-type brane; we refrain from giving explicit expressions here.

3.2 B-type boundary states

Along the same lines as above, one can determine π-permuted B-type boundary states for Gepner models. The A-type Ishibashi conditions (3.2) and (3.3) are replaced by

$$ l_j = l_{\pi(j)}\,, \quad -m_j \equiv m_{\pi(j)} + b_0 \pmod{2k+4}\,, \quad -s_j \equiv s_{\pi(j)} + b_0 + 2b_{\pi(j)} \pmod 4 \tag{3.12} $$

and

$$ -s_0 \equiv s_0 + b_0 + 2\sum_j b_j \pmod 4\,. \tag{3.13} $$

The condition on the m_j implies $m_{\pi^l(j)} = m_{\pi^{l+2}(j)}$ for all l, thus there are at most two independent m_j-values per π-cycle; more precisely

$$ \begin{array}{rlll} 2m_j \equiv -b_0 & \pmod{2k+4} & \text{for all} \quad j \in C^\pi_\nu & \text{if } \Lambda_\nu \text{ is odd}\,, \\ m_{\pi(j)} \equiv -m_j - b_0 & \pmod{2k+4} & \text{for all} \quad j \in C^\pi_\nu & \text{if } \Lambda_\nu \text{ is even}\,. \end{array} $$

Similarly, the constraints on the s-labels require that b_0 is even and that

$$ \begin{array}{ll} 2\sum\limits_{n\in C^\pi_\nu} b_n \equiv 0 \pmod 4 & \text{if } \Lambda_\nu \text{ even}\,, \\ 2s_j + 2\sum\limits_{n\in C^\pi_\nu} b_n \equiv -b_0 \pmod 4 & \text{for all } j \in C^\pi_\nu \quad \text{if } \Lambda_\nu \text{ odd}\,. \end{array} $$

It is straightforward to work out a list of admissible permuted B-type Ishibashi states for the $(3)^5$ model. As before, s_0 and the five s_j-values are only restricted by the β-constraints, and we have the following label structure:

$$ \begin{array}{ll} \pi = \text{id}: & I = (s_0; l_1, l_2, l_3, l_4, l_5; -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_2, -b'_0{+}5\alpha_3, -b'_0{+}5\alpha_4, -b'_0{+}5\alpha_5; s_j) \\ \pi = (1)(2)(3)(4\,5): & I = (s_0; l_1, l_2, l_3, l_4, l_4; -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_2, -b'_0{+}5\alpha_3, \overline{m}_4, -\overline{m}_4{-}2b'_0; s_j) \\ \pi = (1)(2\,3)(4\,5): & I = (s_0; l_1, l_2, l_2, l_3, l_3; -b'_0{+}5\alpha_1, \overline{m}_2, -\overline{m}_2{-}2b'_0, \overline{m}_3, -\overline{m}_3{-}2b'_0; s_j) \\ \pi = (1)(2)(3\,4\,5): & I = (s_0; l_1, l_2, l_3, l_3, l_3; -b'_0{+}5\alpha,1, -b'_0{+}5\alpha_2, -b'_0{+}5\alpha_3, -b'_0{+}5\alpha_3, -b'_0{+}5\alpha_3; s_j) \\ \pi = (1)(2\,3\,4\,5): & I = (s_0; l_1, l_2, l_2, l_2, l_2; -b'_0{+}5\alpha_1, \overline{m}_2, -\overline{m}_2{-}2b'_0, \overline{m}_2, -\overline{m}_2{-}2b'_0; s_j) \\ \pi = (1\,2)(3\,4\,5): & I = (s_0; l_1, l_1, l_2, l_2, l_2; \overline{m}_1, -\overline{m}_1{-}2b'_0, -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_1; s_j) \\ \pi = (1\,2\,3\,4\,5): & I = (s_0; l_1, l_1, l_1, l_1, l_1; -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_1, -b'_0{+}5\alpha_1; s_j) \end{array} \tag{3.14} $$

[1] I am indebted to Matthias Gaberdiel for a crucial discussion of this point.

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Here, the $\overline{m}_i$ and b_0' range over $0, 1, \ldots, 9$, while $\alpha_i = 0, 1$.

Permutation B-type boundary states can be constructed with coefficients very similar to those in (3.7):

$$B^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\alpha}_{A,\pi}} = (-1)^{\frac{s_0^2}{2}} e^{-i\pi\frac{s_0 S_0}{2}} e^{-\frac{i\pi}{2}\sum_{j=1}^5 s_j S_j} \left[\prod_{\nu=1}^{P^\pi} \frac{\sin\pi\frac{(l_\nu+1)(L_\nu+1)}{5}}{\left(\sin\pi\frac{l_\nu+1}{5}\right)^{\Lambda_\nu/2}}\right] \prod_{j=1}^{5} e^{i\pi\frac{m_j M_j}{5}} \tag{3.15}$$

where the m_j are to be expressed by b_0, α_i and $\overline{m}_i$ as in the list above. The labels S_0, S_j and L_ν, $\nu = 1, \ldots, P^\pi$, are as for A-type gluing conditions. Closer inspection shows that B-type boundary states depend only on L_ν and S_j and the following combinations of the five M_j:

$$\begin{aligned} M &:= \sum_{j=1}^{5} M_j \pmod{10}\,, \\ M_{[\nu]} &:= M_{n_\nu^\pi} - M_{\pi(n_\nu^\pi)} + - \ldots - M_{\pi^{\Lambda_\nu-1}(n_\nu^\pi)} \pmod{10} \quad \text{if } \Lambda_\nu \text{ even}\,; \end{aligned} \tag{3.16}$$

here, n_ν^π denotes a chosen representative of the cycle C_ν^π as in (2.4). Thus, π-permuted B-type boundary states with π from the conjugacy classes in the first, fourth and last row of (3.14) are distinguished by the single label M (together with L_ν and S_j of course), while for the other conjugacy classes one also has to specify the values of the alternating M_j-sums over even length cycles. We have the constraints $L_\nu + M_j + S_j = 0 \pmod 2$ for all $j \in C_\nu^\pi$, as well as the identification $(L_\nu, M, M_{[\nu]}, S_j) \equiv (k - L_\nu, M+5, M_{[\nu]}+5, S_j+2)$.

The label structure of permuted B-type branes, which differs from what occurred in the A-type cases, is also reflected in the formulae for B-type partition functions and intersection forms. One finds the following result:

$$\begin{aligned} Z^B_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}(q) \sim \sum_{\boldsymbol{\lambda}',\boldsymbol{\mu}'}^{\text{ev}} \sum_{\rho_0=0}^{19} f_{s,\rho}\,\delta^{(10)}_{5\rho_0+M-\tilde{M}+\Sigma_j m_j'} \left[\prod_{\nu=1}^{P^\pi}\left(\prod_{n\in C_\nu^\pi} N_{l_n'}\right)_{L_\nu\tilde{L}_\nu}\right] \times \\ \times \left[\prod_{\substack{\nu=1\\ \Lambda_\nu \text{odd}}}^{P^\pi} \delta^{(2)}_{\rho_0+M_\nu-\tilde{M}_\nu+\Sigma_{n\in C_\nu^\pi} m_n'}\right] \left[\prod_{\substack{\nu=1\\ \Lambda_\nu\ \text{even}}}^{P^\pi} \delta^{(10)}_{M_{[\nu]}-\tilde{M}_{[\nu]}+m'_{[\nu]}}\right] \chi^{\boldsymbol{\lambda}'}_{\boldsymbol{\mu}'}(q) \end{aligned} \tag{3.17}$$

where we have used the abbreviation $m'_{[\nu]}$ for the alternating sum of m_n' over even cycles in analogy to $M_{[\nu]}$ in (3.16). The summation in the first Kronecker symbol runs over all $j = 1, \ldots, 5$.

The intersection form for B-type boundary conditions $\boldsymbol{\alpha}_\pi$ and $\tilde{\boldsymbol{\alpha}}_\pi$ associated with the same permutation π reads

$$I^B_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi} \sim \sum_{m_1',\ldots,m_5'} \delta^{(10)}_{5+M-\tilde{M}+\Sigma_j m_j'} \prod_{\nu=1}^{P^\pi}\left(\prod_{n\in C_\nu^\pi} N_{m_n'-1}\right)_{L_\nu\tilde{L}_\nu} \prod_{\substack{\nu=1\\ \Lambda_\nu\ \text{even}}}^{P^\pi} \delta^{(10)}_{M_{[\nu]}-\tilde{M}_{[\nu]}+m'_{[\nu]}}\,. \tag{3.18}$$

The summation index ρ_0 would appear only via the combination $5(2\rho_0+1)$ in the $\delta^{(10)}$, thus drops out.

As before, one can compute partition functions and intersection forms between unpermuted and π-permuted B-type branes, and the observations on even cycle length Λ_ν from subsection 3.1 again apply.

4. Open problems

We have considered tensor products of rational CFTs and studied boundary conditions governed by gluing automorphisms from the permutation group. We have presented an ansatz for the associated permutation boundary states and checked that cluster and Cardy's conditions are satisfied. We were able to write down explicit expressions for the open string spectra $Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q)$ for the cases $\pi = \sigma$ and $\pi \neq \sigma = \mathrm{id}$. As a new feature compared to tensor products of ordinary Cardy branes, cycle-wise products of fusion matrices show up in these partition functions. By making use of decompositions of permutations more cleverly, it should be possible to go beyond the non-constructive integrality proof of appendix A and find closed formulae for the multiplicities $n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}$ in the partition functions for arbitrary π and σ.

Whenever $\pi \neq \sigma$, the partition function involves characters of twisted representations. Such characters also play a major role in the recent work [15], where twisted boundary conditions for WZW models were studied; as in the present paper, the gluing conditions are not "aligned" with the automorphism that determines the bulk partition function. In [15], the multiplicities in the open string partition function were expressed in terms of S-matrix elements of ordinary and twined characters, and it would be interesting to compare these expressions to the formulae given here, in the cases covered by both points of view; the results of [16] should prove useful in the process. (A special example occurs in [31], where branes in an asymmetric torus orbifold connected with the 5-fold tensor product of $\widehat{\mathrm{su}}(2)_1$ are studied. The authors could in particular show that the prescriptions given here and in [15], suitably adapted to the non-diagonal partition function of this model, indeed lead to the same boundary states.)

We have not been able to say too much about boundary OPEs and the associativity constraints they must satisfy. Investigating the relation of conformal blocks in cyclic orbifolds to those from the component theory, in particular the results of [32] should be relevant here. The one, very simple brane for which we could study the boundary OPE without high-powered techniques revealed an intriguing connection between (component theory) closed string interactions and interactions of (tensor product) brane excitations. It would be interesting to relate this to the findings of [33], where new connections between boundary conditions and structures in the bulk were uncovered.

In the second part of the paper, we have presented permutation branes for the quintic at the Gepner point. We have given formulae for open string spectra and intersection forms and prepared the ground for a geometric interpretation of the new branes by providing expressions of for some $I_{\boldsymbol{\alpha}_\pi \tilde{\boldsymbol{\alpha}}_\pi}$ in terms of "quantum symmetry" generators. Obviously, such expressions should be found for the missing cases (large cycle lengths and $\pi \neq \sigma$), and one should systematically compute topological invariants of the associated bundles (for B-type branes), following the lines of [24, 25].

But even without a detailed analysis, the intersection forms for B-type permutation branes listed in appendix B suggest that among the boundary states for $\pi = (1)\,(2)\,(3)\,(4\,5)$ there is one with the charges of a configuration made up from D-branes only. (Note that this does not contradict the arguments given in [34], which assume $\pi = \mathrm{id}$.) The charges of some of the new branes will be sums of charges of the "old" boundary states from [23], suggesting that they can be seen as bound states. This might provide tests for some of the conjectures arising from the derived category picture of B-type Calabi-Yau branes developed by Douglas [26].

The A-type permutation branes, on the other hand, can perhaps be exploited for a construction of new special lagrangian cycles for the quintic, using the linear sigma model methods uncovered in [35, 36] and developed further in [37].

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A. Proof of Cardy's conditions for arbitrary permutations

Here we fill the gap left in subsection 2.3 and prove that, in a tensor product of diagonal rational CFTs, the partition functions $Z_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma}(q)$ obey Cardy's conditions for arbitrary permutations $\pi, \sigma \in S_N$. This means we have to show that the coefficients

$$n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi \boldsymbol{\beta}_\sigma} = \sum_{i_{n^*_\lambda} \in \mathcal{I}} \; \prod_{n^\pi_\nu \in C^{\pi*\sigma}_\lambda} B_{\alpha_\nu i_{n^*_\lambda}} \prod_{n^\sigma_\mu \in C^{\pi*\sigma}_\lambda} B_{\beta_\mu i_{n^*_\lambda}} \prod_{n^{\pi^{-1}\sigma}_\rho \in C^{\pi*\sigma}_\lambda} S_{j_{n_\rho} i_{n^*_\lambda}} \tag{A.1}$$

from eq. (2.18) are positive integers. Here, n^*_λ is a fixed representative of the $\lambda^{\rm th}$ orbit $C^{\pi*\sigma}_\lambda$ of $\pi * \sigma$, and for the representatives n^π_ν, n^σ_μ and $n_\rho \equiv n^{\pi^{-1}\sigma}_\rho$ of π-, σ- resp. $(\pi^{-1}\sigma)$-cycles which intersect $C^{\pi*\sigma}_\lambda$, we have put $i_{n^\pi_\nu} = i_{n^*_\lambda}$ etc., implementing the Kronecker symbols in (2.12). (Note that, since we fixed the representatives n^π_ν etc. once and for all, the first product simply runs over all π-cycles that are contained in the $\lambda^{\rm th}$ orbit $C^{\pi*\sigma}_\lambda$ of the group $\pi * \sigma$.)

The coefficients B in (A.1) contain half-integer powers of $S_{0\,i_{n^*_\lambda}}$ in the denominator. From the definition of $C^{\pi*\sigma}_\lambda$, it is however clear that

$$\sum_{\nu:\, n^\pi_\nu \in C^{\pi*\sigma}_\lambda} \Lambda^\pi_\nu = \sum_{\mu:\, n^\sigma_\mu \in C^{\pi*\sigma}_\lambda} \Lambda^\sigma_\mu = \Lambda^*_\lambda\,,$$

the cardinality of $C^{\pi*\sigma}_\lambda$. Thus, the denominator in (A.1) is simply $(S_{0\,i_{n^*_\lambda}})^{\Lambda^*_\lambda}$.

In order to show integrality of $n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi\boldsymbol{\beta}_\sigma}$, let us first assume that $C^{\pi*\sigma}_\lambda$ coincides with a single cycle (the $\lambda^{\rm th}$, say) of π, σ and $\pi^{-1}\sigma$, and that $\Lambda^*_\lambda = 2M+1$ is odd. (E.g., $\pi = (1\,2\,3)$, $\sigma = \pi^{-1}$.) We introduce the objects

$$N^{(M)\;c}_{ab} := \sum_i \frac{S_{a\,i}\,S_{b\,i}\,S_{c\,i}}{\big(S_{0\,i}\big)^{2M+1}} \tag{A.2}$$

such that, for our simplified situation, $n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi\boldsymbol{\beta}_\sigma} = N^{(M)\,j_\lambda}_{\alpha_\lambda\beta_\lambda}$. For $M = 0$, Verlinde's formula tells us that the numbers in (A.2) are the fusion rules. For arbitrary $M \in \mathbb{Z}_+$, they can be rewritten[2] as the numbers of chiral 3-point blocks on a Riemann surface of genus M,

$$\begin{aligned} N^{(M)\;c}_{ab} &= \sum_{\substack{i_1,\dots,i_{2M+1}\\ j_1,\dots,j_{M-1}}} N^{a}_{i_1\,i_2}\, N^{b}_{i_2\,i_3}\, N^{c}_{i_3\,i_4} \prod_{l=1}^{M-1} N^{j_l}_{i_{l+3}\,i_{l+4}}\, N^{j_{M-l}}_{i_{M+2+l}\,i_{M+3+l}} \\ &= \sum_{j_1,\dots,j_{M-1}} \mathrm{tr}\Big(N_a N_b N_c\, N^2_{j_1}\cdots N^2_{j_{M-1}}\Big) \end{aligned} \tag{A.3}$$

where in the first line one identifies $i_{2M+2} = i_1$.

Alternatively, $N^{(M)\;c}_{ab} \in \mathbb{Z}$ follows from the (affine graded fusion ring) relations

$$N^{(M)}_a\, N^{(L)}_b = \sum_c N^{(K)\;c}_{ab}\, N^{(M+L-K)}_c \quad , \tag{A.4}$$

which hold for all $0 \le K \le M+L$; we have used the matrix notation familiar from the fusion rules. Equation (A.4) is easily derived from unitarity of the S-matrix and the representation property (2.9). This shows that the $N^{(M)}_a$ span a commutative associative ring, which moreover is generated by the fusion rules $N^{(0)}_a$ and a single additional matrix $N^{(1)}_0$ with integer entries

$$N^{(1)\;b}_{0\;a} = \sum_i \frac{S_{a\,i}\,S_{b\,i}}{\big(S_{0\,i}\big)^2} = \sum_k N^k_{ab}\,\mathrm{Tr}\,N_k\,.$$

Now let π and σ be "in general position". Then $C^{\pi*\sigma}_\lambda$ covers K^π_λ cycles of π, K^σ_λ cycles of σ and $K^{\pi^{-1}\sigma}_\lambda$ cycles of $\pi^{-1}\sigma$, and the numerator of $n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi\boldsymbol{\beta}_\sigma}$ contains a product of $K_\lambda := K^\pi_\lambda + K^\sigma_\lambda + K^{\pi^{-1}\sigma}_\lambda$ matrix elements of S. But as long as

$$D_\lambda(\pi,\sigma) := \Lambda^*_\lambda - K_\lambda + 3 \;\in\; 2\mathbb{Z}_+ + 1\,, \tag{A.5}$$

we can apply the relation (2.9) repeatedly ($K_\lambda - 3$ times) to reduce to the situation above, and the coefficients $n^{(\lambda)\,J}_{\boldsymbol{\alpha}_\pi\boldsymbol{\beta}_\sigma}$ are products of fusion matrices with some $N^{(M)}_a$, showing that Cardy's constraints are satisfied for arbitrary π and σ.

That the relation (A.5) always holds can be proved by induction in Λ^*_λ, starting from $\Lambda^*_\lambda = 2$, where it is easy to check all possible cases. Now assume that (A.5) is true for all permutations π,σ such that the $C^{\pi*\sigma}_\lambda$ have length (at most) $\Lambda^* - 1$. Obviously, it is

[2] I am grateful to Jean-Bernard Zuber for pointing this out to me.

sufficient to focus on situations where $\pi * \sigma$ has just a single orbit $C_1^{\pi*\sigma}$ of that maximal length, so we can assume that $\pi, \sigma \in S_{\Lambda^*-1}$. In order to increase the orbit length by one, we have to pass to $\tilde{\pi}, \tilde{\sigma} \in S_{\Lambda^*}$. But with the help of transpositions $\tau_{i,j}$, every such permutation can be written as

$$\tilde{\pi} = \hat{\pi} \circ \tau_{i^\pi,\Lambda^*} \qquad \text{for some } \ \pi \in S_{\Lambda^*-1} \qquad \text{and some} \qquad i^\pi \in \{1, \dots, \Lambda^*\}$$

where $\hat{\pi}$ denotes the trivial extension of π to $\{1, \dots, \Lambda^*\}$, i.e. $\hat{\pi}(j) = \pi(j)$ for $1 \leq j \leq \Lambda^* - 1$ and $\hat{\pi}(\Lambda^*) = \Lambda^*$. Analogously, we can write $\tilde{\sigma} = \hat{\sigma} \circ \tau_{i^\sigma,\Lambda^*}$. Except for the trivial case $i^\pi = i^\sigma = \Lambda^*$, we have $C_1^{\tilde{\pi}*\tilde{\sigma}} = \Lambda^*$. If $i^\pi \neq \Lambda^*$ and $i^\pi \in C_\nu^\pi$, then the ν^{th} cycle of $\tilde{\pi}$ is obtained from the ν^{th} cycle of π by placing Λ^* right behind i^π, and $K_1^{\tilde{\pi}} = K_1^\pi$. If $i^\pi = \Lambda^*$, then $K_1^{\tilde{\pi}} = K_1^\pi + 1$, but $K_1^{\tilde{\sigma}} = K_1^\sigma$ since $i^\sigma \neq \Lambda^*$. The permutation $\pi^{-1}\sigma$ changes to $\tilde{\pi}^{-1}\tilde{\sigma} = \tau_{i^\pi,\Lambda^*} \circ \pi^{-1}\sigma \circ \tau_{i^\sigma,\Lambda^*}$. If $i^\sigma \neq \Lambda^*$, then Λ^* is inserted in an existing $(\pi^{-1}\sigma)$-cycle directly behind i^σ. The effect of τ_{i^π,Λ^*}, assuming $i^\pi \neq \Lambda^*$, depends on whether i^π is an element of that $(\pi^{-1}\sigma)$-cycle or not: In the first case, the cycle is split into two (between i^π and its predecessor in the cycle); in the second case, the $(\pi^{-1}\sigma)$-cycle containing i^π is joined with the one containing i^σ. Counting the number of $\tilde{\pi}$-, $\tilde{\sigma}$- and $(\tilde{\pi}^{-1}\tilde{\sigma})$-cycles covered by $C_1^{\tilde{\pi}*\tilde{\sigma}}$, it is easy to see that

$$D_1(\tilde{\pi}, \tilde{\sigma}) = D_1(\pi, \sigma) \quad \text{or} \quad D_1(\pi, \sigma) + 2$$

for all possible i^π, i^σ and cycle structures — so $D_1(\tilde{\pi}, \tilde{\sigma})$ is odd as required. All in all, we have shown that (A.5) indeed holds and therefore that all pairs of boundary states defined by (2.2) and (2.6) satisfy Cardy's conditions.

B. Some intersection forms

We express some of the intersection forms for permutation branes on the quintic in terms of charge symmetry generators. The fields and states of bulk Gepner models transform under a discrete charge symmetry group which, up to $\mathbb{Z}_2$-factors we will ignore, is given by $(\mathbb{Z}_5)^4$ in the case of the quintic. The generators act on Ishibashi states (for both A- and B-type gluing conditions) as follows:

$$G_j |\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle\rangle = e^{\frac{2\pi i}{5} m_j} \, |\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle\rangle$$

for $j = 1, \dots, 5$. Since the β-constraints imply that the total charge of each state is integer, one has the relation $G_1 \, G_2 \, G_3 \, G_4 \, G_5 = 1$. On boundary states, the $\mathbb{Z}_5$-generators act by shifting the M-labels; we denote the generator acting on the label M_ξ by g_ξ, where the index ξ may stand for the number ν of a π-cycle, for $[\nu]$ in the B-type cases, etc. For π-permuted A-type branes, we have

$$\begin{aligned} g_\nu : \ M_\nu &\longmapsto M_\nu + 2 \qquad \text{for } \ P^\pi > 1\,, \\ g : M &\longmapsto M + 2\,, \qquad g' : M' \longmapsto M' + 1 \qquad \text{for} \quad P^\pi = 1\,. \end{aligned}$$

The two labels M and M', and likewise the generators g and g', are completely independent for $\pi = (1\,2\,3\,4\,5)$, while for $P^\pi > 1$ the β-constraints enforce the relation

$$g_1^{\Lambda_1^\pi}\, g_2^{\Lambda_2^\pi} \cdots g_{P^\pi}^{\Lambda_{P^\pi}^\pi} = 1 \quad . \tag{B.1}$$

For B-type boundary states, the labels M and $M_{[\nu]}$ are again independent, and so are g and $g_{[\nu]}$, acting as

$$g : M \longmapsto M + 2\,, \qquad g_{[\nu]} : M_{[\nu]} \longmapsto M_{[\nu]} + 2\,,.$$

Since the intersection forms $I_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}$ depend only on differences of the M_ξ- and $\tilde{M}_\xi$-variables, they can be written in terms of $(\mathbb{Z}_5)^R$-generators with

$$\begin{array}{lll} \text{A-type:} & R = P^\pi - 1 & \text{for} \quad P^\pi > 1 \\ & R = 2 & \text{for} \quad P^\pi = 1 \\ \text{B-type:} & R = 1 + P^\pi_{\text{ev}} & \end{array}$$

with P^π_{ev} being the number of even length cycles of the permutation π. The charge symmetry is therefore different from the $\pi = \text{id}$ case analysed in [24].

For $\pi = \text{id}$ branes, closed formulae for intersection forms in terms of symmetry generators are easy to write down once the SU(2) fusion rules have been expressed through the g_ν. In our more general case, analogous expressions have to be found for various products of fusion matrices — a task that becomes more and more tedious as the cycle lengths increase. We therefore give formulae for A-type intersection forms $I^A_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}$ in terms of $\mathbb{Z}_5$-generators only for permutations with cycles of length up to $\Lambda_\nu = 4$. For $L_\nu = \tilde{L}_\nu = 0$, the intersection forms are

$$\begin{aligned}
\pi &= \text{id}: \\
&\quad I \sim (1-g_1)(1-g_2)(1-g_3)(1-g_4)(1-g_5) \\
\pi &= (1)(2)(3)(4\,5): \\
&\quad I \sim (1-g_1)(1-g_2)(1-g_3)(1-g_4)(1-3g_4-2g_4^2-g_4^3) \\
\pi &= (1)(2\,3)(4\,5): \\
&\quad I \sim (1-g_1)(1-g_2)(1-3g_2-2g_2^2-g_2^3)(1-g_3)(1-3g_3-2g_3^2-g_3^3) \\
\pi &= (1)(2)(3\,4\,5): \\
&\quad I \sim (1-g_1)(1-g_2)(1-g_3)(1+3g_3+g_3^2) \\
\pi &= (1)(2\,3\,4\,5): \\
&\quad I \sim (1-g_1)(1-g_2^2)^2 \\
\pi &= (1\,2)(3\,4\,5): \\
&\quad I \sim (1-g_1)(1-3g_1-2g_1^2-g_1^3)(1-g_2)(1+3g_2+g_2^2)\,.
\end{aligned}$$

In each line, one can eliminate one of the group generators by means of (B.1). Computing $I_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}$ for higher L-values, one finds the same behaviour as in [24]: for each L_ν or $\tilde{L}_\nu$ that is raised from 0 to 1, the g_ν-dependent factor has to be multiplied by $(g_\nu^{1/2} + g_\nu^{-1/2})$.

For B-type permutation branes, the intersection forms $I^B_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}$ have the following form — up to cycle length $\Lambda_\nu = 3$, and again with $L_\nu = \tilde{L}_\nu = 0$:

$$
\begin{array}{ll}
\pi = \text{id}: & I \sim\ (1-g)^5 \\
\pi = (1)(2)(3)(4\,5): & I \sim\ g(1-g)^3 \cdot N\left(g_{[4]}\right) \\
\pi = (1)(2\,3)(4\,5): & I \sim\ g^2(1-g) \cdot N\left(g_{[2]}\right) \cdot N\left(g_{[3]}\right) \\
\pi = (1)(2)(3\,4\,5): & I \sim\ g^2(1-g) \\
\pi = (1\,2)(3\,4\,5): & I \sim\ g(1-g)(1+3g+g^2) \cdot N\left(g_{[1]}\right).
\end{array}
$$

We have used the abbreviation $N(g_{[\nu]}) := 1 + g_{[\nu]} + g^2_{[\nu]} + g^3_{[\nu]} + g^4_{[\nu]}$ for the $g_{[\nu]}$-dependent contribution to the intersection forms. The form of $N(g_{[\nu]})$ simply means that the intersection of two boundary states depends only on their M-labels and not on the labels $M_{[\nu]}$ associated with cycles of even length; cf. to the analogous observation made in [29] in the context of torsion branes.

In order to obtain intersection forms for boundary states where an L_ν or $\tilde{L}_\nu$ is raised from 0 to 1, one multiplies the g-dependent part by $(g^{\frac{1}{2}} + g^{-\frac{1}{2}})$ and, if Λ_ν is even, $N(g_{[\nu]})$ by $(g^{\frac{1}{2}}_{[\nu]} + g^{-\frac{1}{2}}_{[\nu]})$.

One observes that g-dependent factors in these B-type intersection forms I^B_π (with all L-labels zero) can be obtained from the A-type intersection forms I^A_π for the same permutation π by replacing all the g_i by the single $\mathbb{Z}_5$-generator g, up to an overall multiplicity which we left undetermined anyway. Thus, the natural conjecture for $I^B_{\boldsymbol{\alpha}_\pi\,\tilde{\boldsymbol{\alpha}}_\pi}$ with $\pi = (1)(2\,3\,4\,5)$ is

$$
I \sim g(1-g)(1+3g+g^2)
$$

up to a (probably again irrelevant) factor depending on $g_{[2]}$. This could of course be checked directly, by starting from (3.18) and going through some rather tedious combinatorics. Note that, up to the factors involving $g_{[\nu]}$, the B-type intersection forms $I_{\pi,\pi}$ depend *only on the number of cycles of* π.

One can now in principle follow the methods of [24] and compare the g-parts of I^π_B to the geometric B-type intersection form I^{geo}_B of even-dimensional cycles, which at the Gepner point is given by

$$
I^{\text{geo}}_B = -g(1-g)^3\,,
$$

see [24]. Up to overall normalisation, we find that $I^\pi_B \sim m_\pi(g)\; I^{\text{geo}}_B\; m_\pi(g^{-1})$ with $m_\pi(g) = 1 - g$ for $\pi = \text{id}$ as in [24] and $m_\pi(g) = 1$ for $\pi = (1)\,(2)\,(3)\,(4\,5)$.

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